Warsaw University of Technology

Faculty of Power and Aeronautical Engineering

WARSAW UNIVERSITY OF TECHNOLOGY

Institute of Aeronautics and Applied Mechanics

Finite element method (FEM)

Requirements for the shape functions

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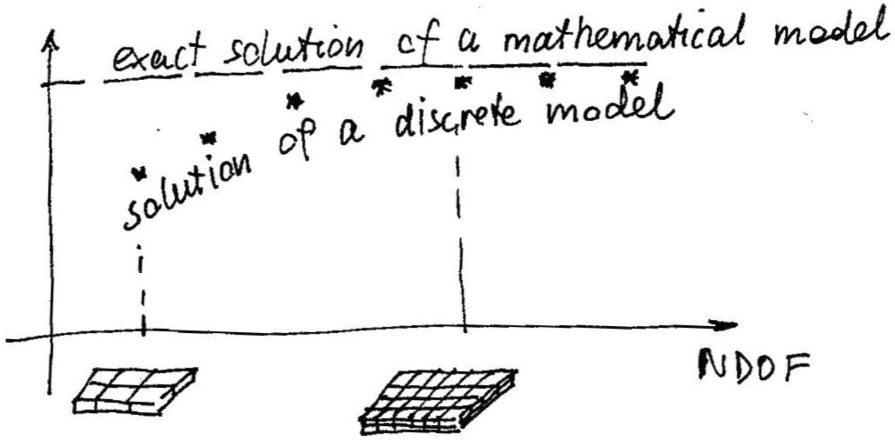
Requirements for the shape functions

a) allow approximation of the constant value of the function $\{u\}$ inside the finite element,

b) ensure continuity on the border between finite elements of the function $\{u\}$ and its derivatives up to one order smaller than the highest derivative of $\{u\}$ existing in the functional of the total potential energy V.

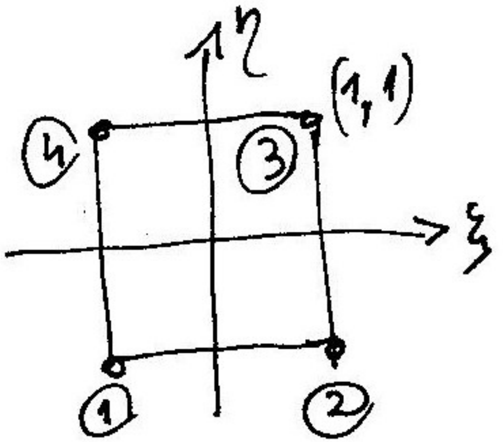
If requirements a) and b) are satisfied, then the approximate solution tends to the exact solution when increasing the number of degrees of freedom.



Example. Check requirements for the shape functions of 4-node quadrilateral element.

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta), \quad N_2 = \frac{1}{4}(1+\xi)(1-\eta), \quad N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

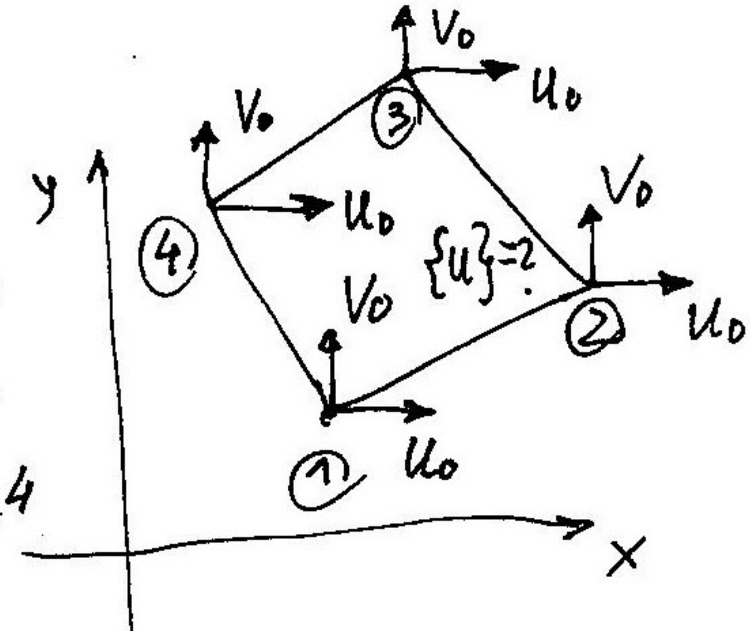


a)

$u_i = u_0$

$v_i = v_0$

$i = 1,2,3,4$



$$\underset{2\times1}{\{u\}} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \cdot \begin{Bmatrix} u_0 \\ v_0 \\ u_0 \\ v_0 \\ u_0 \\ v_0 \\ u_0 \\ v_0 \end{Bmatrix} =$$

$$= \begin{Bmatrix} (N_1 + N_2 + N_3 + N_4)\, u_0 \\ (N_1 + N_2 + N_3 + N_4)\, v_0 \end{Bmatrix} =$$

$$= \begin{Bmatrix} \frac{1}{4}\left((1-\xi)\cdot(1-\eta) + (1+\xi)(1-\eta) + (1+\xi)(1+\eta) + (1-\xi)(1+\eta) \right) \cdot u_0 \\ \frac{1}{4}(\dots)\, v_0 \end{Bmatrix} =$$

$$= \begin{Bmatrix} \frac{1}{4}\left((1-\xi+1+\xi)(1-\eta) + (1+\xi+1-\xi)(1+\eta) \right) u_0 \\ \frac{1}{4}(\dots)\, v_0 \end{Bmatrix} =$$

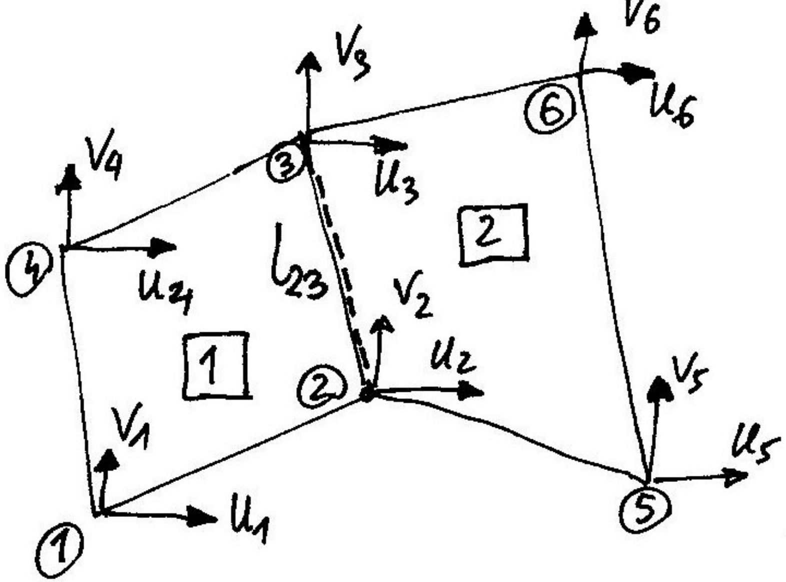
$$= \begin{Bmatrix} \frac{1}{4}\left(2(1-\eta) + 2(1+\eta) \right) u_0 \\ \frac{1}{4}(\dots)\, v_0 \end{Bmatrix} = \begin{Bmatrix} u_0 \\ v_0 \end{Bmatrix} \quad a) - \text{satisfied}$$

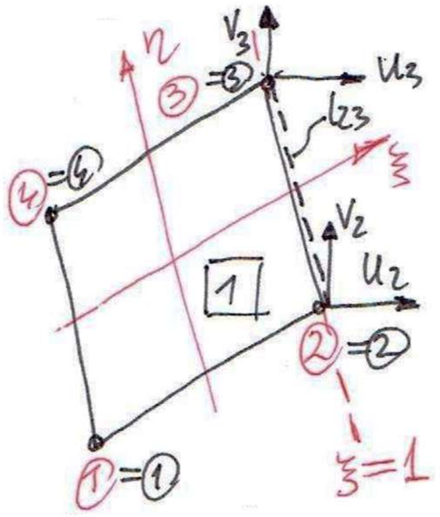
b) $V_e = U_e - W_e = \frac{1}{2}\int\limits_{\Omega e} \underset{1\times3}{\lfloor \varepsilon \rfloor} \underset{3\times3}{[D]} \underset{3\times1}{\{\varepsilon\}} \, d\Omega e - \int\limits_{\Omega e} \underset{1\times2}{\lfloor X \rfloor} \underset{2\times1}{\{u\}} \, d\Omega e - \int\limits_{\Gamma_{pe}} \underset{1\times2}{\lfloor p \rfloor} \underset{2\times1}{\{u\}} \, d\Gamma_{pe}$

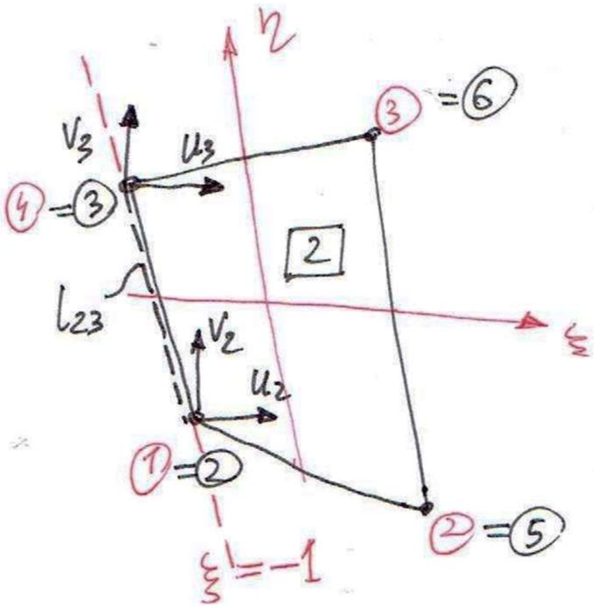
$\lfloor \varepsilon \rfloor$ $\|$ $\underset{3\times2}{[R]} \underset{2\times1}{\{u\}}$ $\Rightarrow$ (the first order is the highest order of derivatives in the functional V.)

$[R]$: includes differential operators of the first order

The requirement b) is satisfied if the function $\{u\}$ is continous between finite elements







shape functions at l_{23} :

$$N_1 = 0,\ N_2 = \tfrac{1}{2}(1-\eta),\ N_3 = \tfrac{1}{2}(1+\eta),\ N_4 = 0$$

$$u\Big|_{23}^{\boxed{1}} = N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4 = $$
$$= N_2u_2 + N_3u_3 = \tfrac{1}{2}\left((1-\eta)u_2 + (1+\eta)u_3\right)$$

$$v\Big|_{23}^{\boxed{1}} = \tfrac{1}{2}\left((1-\eta)v_2 + (1+\eta)v_3\right)$$

shape functions at l_{23} :

$$N_1 = \tfrac{1}{2}(1+\eta),\ N_2 = 0,\ N_3 = 0,\ N_4 = \tfrac{1}{2}(1+\eta)$$

$$u\Big|_{23}^{\boxed{2}} = N_1\cdot u_2 + N_2\cdot u_5 + N_3\cdot u_6 + N_4\cdot u_3 = $$
$$= N_1\cdot u_2 + N_4\cdot u_3 = \tfrac{1}{2}\left((1-\eta)u_2 + (1+\eta)u_3\right)$$

$$v\Big|_{23}^{\boxed{2}} = \tfrac{1}{2}\left((1-\eta)v_2 + (1+\eta)v_3\right)$$

$$u\Big|_{23}^{\boxed{1}} = u\Big|_{23}^{\boxed{2}} \quad \text{and} \quad v\Big|_{23}^{\boxed{1}} = v\Big|_{23}^{\boxed{2}} \implies \text{b) - satisfied}$$